

# MONOTONICITY FORMULAS IN POTENTIAL THEORY

VIRGINIA AGOSTINIANI AND LORENZO MAZZIERI

**ABSTRACT.** Using the electrostatic potential  $u$  due to a uniformly charged body  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , we introduce a family of monotone quantities associated with the level set flow of  $u$ . The derived monotonicity formulas are exploited to deduce sharp geometric inequalities involving the electrostatic capacity of  $\Omega$  and the mean curvature of its boundary. As a byproduct we also recover the classical Willmore inequality, characterizing the equality case in terms of rotational symmetry.

MSC (2010): 35B06, 53C21, 35N25.

Keywords: electrostatic capacity, overdetermined boundary value problems, Willmore inequality.

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

**1.1. Setting of the problem and statement of the main result.** We consider the electrostatic potential due to a charged body, modeled by a bounded domain  $\Omega$  with  $\mathcal{C}^{2,\alpha}$ -boundary, for some  $0 < \alpha < 1$ . The potential is defined as the unique solution  $u$  of the following problem in the exterior domain

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases} \quad (1.1)$$

Throughout the paper we assume that  $\partial\Omega$  is a regular level set of  $u$ . It is worth pointing out that for every  $0 < t \leq 1$  the level set  $\{u = t\}$  is compact, due to the properness of  $u$ . Moreover, we have that for every  $t > 0$  sufficiently small the level set  $\{u = t\}$  is diffeomorphic to a  $(n-1)$ -dimensional sphere, and thus connected. These properties can be deduced from expansion (1.2) below.

A natural quantity associated with a solution of problem (1.1) is the electrostatic capacity of  $\Omega$ , which is defined as

$$\text{Cap}(\Omega) := \inf \left\{ \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |Dw|^2 d\mu \mid w \in C_c^\infty(\mathbb{R}^n), w \equiv 1 \text{ in } \Omega \right\}.$$

A classical fact (see for instance [20]) is that  $u$ , as well as its first and second order derivatives, satisfies in standard Euclidean coordinates the asymptotic expansions

$$\begin{aligned} u &= \text{Cap}(\Omega) |x|^{2-n} + o(|x|^{2-n}), \\ D_\alpha u &= -(n-2) \text{Cap}(\Omega) |x|^{-n} x_\alpha + o(|x|^{1-n}), \\ D_{\alpha\beta}^2 u &= (n-2) \text{Cap}(\Omega) |x|^{-n-2} (n x_\alpha x_\beta - |x|^2 \delta_{\alpha\beta}) + o(|x|^{-n}), \end{aligned} \quad (1.2)$$

as  $|x| \rightarrow +\infty$ , for every  $1 \leq \alpha, \beta \leq n$ . On the other hand, the capacity of  $\Omega$  can be computed in terms of the electrostatic potential  $u$  as

$$\text{Cap}(\Omega) (n-2) |\mathbb{S}^{n-1}| = \int_{\partial\Omega} |Du| d\sigma = \int_{\{u=t\}} |Du| d\sigma,$$

where in the second equality we have used the Divergence Theorem and the equation  $\Delta u = 0$ . This last fact can be rephrased by saying that the function  $U_1 : (0, 1] \rightarrow \mathbb{R}$  given by

$$t \mapsto U_1(t) := \int_{\{u=t\}} |Du| d\sigma$$

is constant and  $U_1(t) \equiv \text{Cap}(\Omega)(n-2)|\mathbb{S}^{n-1}|$ , for every  $t \in (0, 1]$ . In analogy with this, we introduce for  $p \geq 0$  the functions  $U_p : (0, 1] \rightarrow \mathbb{R}$  given by

$$t \mapsto U_p(t) := \left[ \frac{\text{Cap}(\Omega)}{t} \right]^{\frac{(p-1)(n-1)}{(n-2)}} \int_{\{u=t\}} |Du|^p d\sigma, \quad (1.3)$$

and we observe that using expansion (1.2) one can easily compute the limit

$$\lim_{t \rightarrow 0^+} U_p(t) = [\text{Cap}(\Omega)]^p (n-2)^p |\mathbb{S}^{n-1}|. \quad (1.4)$$

Before proceeding with the statement of the main result, it is worth describing some other features of the functions  $t \mapsto U_p(t)$ . First of all we notice that they are well defined, since the integrands are globally bounded and the level sets of  $u$  have finite hypersurface area. In fact, since  $u$  is harmonic, the level sets of  $u$  have locally finite  $\mathcal{H}^{n-1}$ -measure (see for instance [18] and the references therein). Moreover, by the properness of  $u$ , they are compact and thus their hypersurface area is finite. To describe another important feature of the  $U_p$ 's, we recall that in the case where  $\Omega$  is a ball the explicit solution to problem (1.1) is given by (a multiple of) the Green function. Hence, the expansions reported in (1.2) deprived of the reminder terms yield in this case explicit formulæ for  $u$ ,  $Du$  and  $D^2u$  on the whole  $\mathbb{R}^n \setminus \bar{\Omega}$ . Tacking advantage of this observation, one easily realizes that the quantities

$$\mathbb{R}^n \setminus \bar{\Omega} \ni x \mapsto \frac{|Du|}{u^{\frac{n-1}{n-2}}}(x) \quad \text{and} \quad (0, 1] \ni t \mapsto \int_{\{u=t\}} u^{\frac{n-1}{n-2}} d\sigma \quad (1.5)$$

are constant on rotationally symmetric solutions (notice *en passant* that the square of the first quantity is known in the literature as the  $P$ -function naturally associated with problem (1.1). In a moment we will be able to shade some lights on the geometric nature of such a function). On the other hand, for every  $p \geq 0$ , the functions  $t \mapsto U_p(t)$  can be rewritten in terms of the quantities appearing in (1.5) as

$$U_p(t) = [\text{Cap}(\Omega)]^{\frac{(p-1)(n-1)}{(n-2)}} \int_{\{u=t\}} \left( \frac{|Du|}{u^{\frac{n-1}{n-2}}} \right)^p u^{\frac{n-1}{n-2}} d\sigma. \quad (1.6)$$

Therefore they are constant on rotationally symmetric solutions. In contrast with this, our main result states that the functions  $t \mapsto U_p(t)$  are in general monotonically nondecreasing and that the monotonicity is strict unless  $u$  is rotationally symmetric.

**Theorem 1.1** (Monotonicity-Rigidity Theorem). *Let  $u$  be a solution to problem (1.1) and let  $U_p : (0, 1] \rightarrow \mathbb{R}$  be the function defined in (1.3). Then, the following properties hold true.*

- (i) *For every  $p \geq 1$ , the function  $U_p$  is continuous.*
- (ii) *For every  $p \geq 2 - 1/(n-1)$ , the function  $U_p$  is differentiable and the derivative satisfies, for every  $t \in (0, 1]$ ,*

$$U'_p(t) = (p-1) \left[ \frac{\text{Cap}(\Omega)}{t} \right]^{\frac{(p-1)(n-1)}{(n-2)}} \int_{\{u=t\}} |Du|^{p-1} \left[ H - \left( \frac{n-1}{n-2} \right) |D(\log u)| \right] d\sigma \geq 0, \quad (1.7)$$

where  $H$  is the mean curvature of the level set  $\{u = t\}$  computed with respect to the unit normal vector  $\nu = -Du/|Du|$ . Moreover, if there exists  $t \in (0, 1]$  such that  $U'_p(t) = 0$ , then  $u$  is rotationally symmetric.

It is worth mentioning that monotonicity formulas are nowadays known to play a fundamental role in geometric analysis (dropping any attempt of being complete, we mention [19, 22] and also the more recent [7]), as well as in the study of geometric properties of harmonic functions on manifolds subject to suitable curvature lower bounds [11, 13, 12]. With regard to the last mentioned references, it would be interesting to explore the connections between our formulas and the ones obtained by Colding and coauthors.

**Remark 1.** Notice that formula (1.7) is well-posed also in the case where  $\{u = t\}$  is not a regular level set of  $u$ . Indeed, as already observed, it is well-known that, since  $u$  is harmonic and proper, the  $(n-1)$ -dimensional Hausdorff measure of each of its level sets is finite. Moreover, by the results in [17] (see also [8]), the Hausdorff dimension of its critical set is bounded above by  $(n-2)$ . In particular, the unit normal is well defined  $\mathcal{H}^{n-1}$ -almost everywhere on each level set and so does the mean curvature  $H$ . In turn, the integrand in (1.7) is well defined  $\mathcal{H}^{n-1}$ -almost everywhere. Finally, we observe that where  $|Du| \neq 0$  it holds

$$|Du|^{p-1} H = |Du|^{p-4} D^2u(Du, Du).$$

Since  $|D^2u|$  is uniformly bounded in  $\mathbb{R}^n \setminus \bar{\Omega}$ , this shows that the integrand in (1.7) is essentially bounded and thus summable on every level set of  $u$ , provided  $p \geq 2$ . In the cases where  $2 - 1/(n-1) \leq p < 2$ , it is no longer possible to infer that the function  $|Du|^{p-1} H$  is essentially bounded on the critical level sets of  $u$ . Nevertheless, we can prove that it is still summable (see Remark 5).

**Remark 2.** Notice that under the hypothesis of the above theorem, formula (1.7) implies the non existence of minimal level sets of  $u$  and in particular the non existence of smooth minimal compact hypersurfaces in  $\mathbb{R}^n$ .

**Remark 3.** Let us observe that a perusal of the proof of Theorem 3.2, which in turn gives Theorem 1.1 as explained later on, shows that the function  $U_p$  is continuous and differentiable for every  $p \geq 0$  and in every region  $[t_1, t_2]$ , whenever  $0 < t_1 < t_2 \leq 1$  are such that  $|Du| > 0$  in  $\{t_1 \leq u \leq t_2\}$ .

**1.2. Strategy of the proof.** In this subsection, we present the main ideas underlying the proof of the Monotonicity-Rigidity Theorem. To do this, we focus for simplicity on the case  $p = 3$ . Our strategy consists of two main steps. The first step is the construction of a *cylindrical ansatz*, that is a metric  $g$  conformally equivalent to the Euclidean metric  $g_{\mathbb{R}^n}$  thorough the conformal factor  $u^{\frac{2}{n-2}}$ , namely

$$g = u^{\frac{2}{n-2}} g_{\mathbb{R}^n}.$$

The reason for the name is that when  $u$  is rotationally symmetric, then  $g$  is the cylindrical metric. Before proceeding, we recall that the same strategy described here is at the basis of the results of [1], where in place of the Euclidean metric and of the electrostatic potential we considered a *static metric* together with the associate *static potential*. The cylindrical ansatz leads to a reformulation of problem (1.1) where the new metric  $g$  and the  $g$ -harmonic function  $\varphi = -\log u$  fulfill the quasi-Einstein type equation

$$\text{Ric}_g - \nabla^2 \varphi + \frac{d\varphi \otimes d\varphi}{n-2} = \frac{|\nabla \varphi|_g^2}{n-2} g, \quad \text{in } M.$$

Here,  $M = \mathbb{R}^n \setminus \Omega$  and  $\nabla$  is the Levi-Civita connection of  $g$ . Before proceeding, it is worth pointing out that taking the trace of the above equation gives

$$\frac{R_g}{n-1} = \frac{|\nabla \varphi|_g^2}{n-2},$$

where  $R_g$  is the scalar curvature of the conformal metric  $g$ . Hence,  $|\nabla \varphi|_g^2$  is expected to be constant precisely when  $(M, g)$  is isometric to a round cylinder. Noticing that

$$|\nabla \varphi|_g = \frac{|Du|}{u^{\frac{n-1}{n-2}}} \quad (1.8)$$

and recalling the little discussion after formula (1.5), we obtain a clear geometric interpretation of the constancy of the  $P$ -function associated with problem (1.1).

The second step of our strategy consists in proving via a splitting principle that the metric  $g$  has indeed a product structure, provided the hypothesis of the Rigidity statement is satisfied (splitting techniques have been successfully employed in the context of partial differential equations for example in [2, 16]). More in general, we use the above conformal reformulation of the original system combined with the Bochner identity to deduce the equation

$$\Delta_g |\nabla \varphi|_g^2 - \langle \nabla |\nabla \varphi|_g^2, \nabla \varphi \rangle_g = 2 |\nabla^2 \varphi|_g^2.$$

Observing that the drifted Laplacian appearing on the left hand side is formally self-adjoint with respect to the weighted measure  $e^{-\varphi} d\mu_g$ , we integrate by parts and obtain, for every  $s \geq 0$ , the integral identity

$$\int_{\{\varphi=s\}} \frac{|\nabla \varphi|_g^2 H_g}{e^s} d\sigma_g = \int_{\{\varphi>s\}} \frac{|\nabla^2 \varphi|_g^2}{e^\varphi} d\mu_g, \quad (1.9)$$

where  $H_g$  is the mean curvature of the level set  $\{\varphi = s\}$  inside the ambient  $(M, g)$ . We then observe that, up to a negative function of  $s$ , the left-hand side coincides with  $U'_3$  (see formulæ (3.11) and (3.12) below), whereas the right hand side is always nonnegative. This implies the Monotonicity statement. Also, under the hypothesis of the Rigidity statement, the left hand side of the above identity vanishes and thus the Hessian of  $\varphi$  must be zero in an open region of  $M$ . In turn, by analyticity, it vanishes everywhere. On the other hand, the asymptotic behavior of  $u$  implies that  $\varphi(x) \rightarrow +\infty$  when  $x \rightarrow \infty$ . In particular,  $\nabla \varphi$  is a nontrivial parallel vector field. Hence, it provides a natural splitting direction for the metric  $g$ , which can then be proved to have a product structure. Finally, using the fact that  $g_{\mathbb{R}^n}$  is flat and thus  $g$  is conformally flat by construction, we can argue as in [2] that the cross sections of the Riemannian product  $(M, g)$  are indeed metric spheres and that in turn  $(M, g)$  is isometric to a round cylinder.

For an arbitrary  $p \geq 2 - 1/(n-1)$ , we obtain in place of (1.9) the more general identity (4.8), where the left-hand side coincides with  $U'_p$ , up to a negative function of  $s$ . As for the case  $p = 3$ , the monotonicity and

rigidity results are obtained thanks to the nonnegativity of the right-hand side of (4.8), which is ensured by the standard Kato inequality for  $p \geq 2$  and by the refined Kato inequality for harmonic functions for  $2 - 1/(n-1) \leq p < 2$ .

## 2. CONSEQUENCES OF THE MONOTONICITY-RIGIDITY THEOREM

In this section we discuss some consequences of Theorem 1.1. We start with an integral inequality, which follows directly from formula (1.7). The equality case characterizes the rotationally symmetric solutions to problem (1.1).

**Theorem 2.1.** *Let  $u$  be a solution to problem (1.1). Then, for every  $p \geq 2 - 1/(n-1)$  and every  $t \in (0, 1]$ , the inequality*

$$\int_{\{u=t\}} \frac{|D(\log u)|^p}{n-2} d\sigma \leq \int_{\{u=t\}} |D(\log u)|^{p-1} \frac{H}{n-1} d\sigma \quad (2.1)$$

*holds true, where  $H$  is the mean curvature of the level set  $\{u = t\}$ . Moreover, the equality is fulfilled for some  $p \geq 2 - 1/(n-1)$  and some  $t \in (0, 1]$  if and only if  $u$  is rotationally symmetric.*

In the case  $p = 3$ , this theorem was first proved in [2]. See also [5] for an alternative proof. To give a consequence of Theorem 1.1 in the framework of overdetermined boundary value problems, we observe that the equality is achieved in (1.7) as soon as the term in square brackets vanishes  $\mathcal{H}^{n-1}$ -almost everywhere on some level set of  $u$ . This easily implies the following corollary, which was already pointed out in [2].

**Corollary 2.2.** *Let  $u$  be a solution to problem (1.1) and assume that the identity*

$$\frac{|Du|}{n-2} = \frac{H}{n-1} \quad (2.2)$$

*holds  $\mathcal{H}^{n-1}$ -almost everywhere on  $\partial\Omega$ , where  $H$  is the mean curvature of  $\partial\Omega$ . Then, the solution  $u$  is rotationally symmetric. In particular, if  $u$  solves the overdetermined boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = -\left(\frac{n-2}{n-1}\right) H & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (2.3)$$

*where  $\nu$  is the unit normal vector of  $\partial\Omega$  pointing toward the interior of  $\mathbb{R}^n \setminus \overline{\Omega}$ , then  $\Omega$  is ball and  $u$  is rotationally symmetric.*

In other words, assumption (2.2) in the previous corollary can be seen as a condition that makes the problem (1.1) overdetermined and forces the solution to be rotationally symmetric. Observe that (2.2) is always satisfied on a rotationally symmetric solution and thus it is also a necessary condition for  $u$  being rotationally symmetric. It is worth noticing that the overdetermining condition (2.2) can be phrased by saying that the normal derivative of the solution equals the mean curvature of the boundary, up to a constant factor. On the other hand, we recall that if either the right hand side (mean curvature) or the left hand side (normal derivative) are imposed to be constant, then the rigidity holds, by the celebrated results of Alexandroff [3] and Reichel [23], respectively. It is also interesting to compare the overdetermined boundary value problem (2.3) with the one proposed in [14, Conjecture 10].

To describe other implications of Theorem 2.1, let us observe that, applying Hölder inequality to the right hand side of (2.1) with conjugate exponents  $p/(p-1)$  and  $p$ , one gets

$$\int_{\{u=t\}} |D(\log u)|^{p-1} H d\sigma \leq \left( \int_{\{u=t\}} |D(\log u)|^p d\sigma \right)^{(p-1)/p} \left( \int_{\{u=t\}} |H|^p d\sigma \right)^{1/p}.$$

This implies on every level set of  $u$  the following sharp  $L^p$ -bound for the gradient of  $u$  in terms of the  $L^p$ -norm of the mean curvature of the level set.

**Theorem 2.3.** *Let  $u$  be a solution to problem (1.1). Then, for every  $p \geq 2 - 1/(n-1)$  and every  $t \in (0, 1]$  the inequality*

$$\left\| \frac{D(\log u)}{n-2} \right\|_{L^p(\{u=t\})} \leq \left\| \frac{H}{n-1} \right\|_{L^p(\{u=t\})} \quad (2.4)$$

holds true, where  $H$  is the mean curvature of the level set  $\{u = t\}$ . Moreover, the equality is fulfilled for some  $p \geq 2 - 1/(n - 1)$  and some  $t \in (0, 1]$  if and only if  $u$  is rotationally symmetric.

Setting  $t = 1$  in (2.4), and interpreting  $|D(\log u)| = |Du|$  on  $\partial\Omega$  as the absolute value of the normal derivative, one gets the following corollary.

**Corollary 2.4.** *Let  $u$  be a solution to problem (1.1). Then, for every  $p \geq 2 - 1/(n - 1)$  the inequality*

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^p(\partial\Omega)} \leq \left( \frac{n-2}{n-1} \right) \|H\|_{L^p(\partial\Omega)} \quad (2.5)$$

holds true, where  $H$  is the mean curvature of  $\partial\Omega$  and  $\nu$  is the unit normal vector of  $\partial\Omega$  pointing toward the interior of  $\mathbb{R}^n \setminus \bar{\Omega}$ . Moreover, the equality is fulfilled for some  $p \geq 2 - 1/(n - 1)$  if and only if  $u$  is rotationally symmetric. Finally, letting  $p \rightarrow +\infty$  in the previous inequality, one has that

$$\max_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right| \leq \left( \frac{n-2}{n-1} \right) \max_{\partial\Omega} |H|. \quad (2.6)$$

Concerning inequality (2.6), unfortunately we do not know whether the rigidity statement holds true or not. However, the equality is fulfilled in the case of rotationally symmetric solutions and this makes the inequality sharp.

Recalling that the electrostatic capacity of a charged body  $\Omega$  can be computed in terms of the exterior normal derivative as

$$\text{Cap}(\Omega) = - \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma,$$

and using the Hölder inequality, it is not hard to deduce from (2.5) and (2.6) the following geometric upper bounds for the capacity. Observe that  $\partial u / \partial \nu = -|Du| < 0$  on  $\partial\Omega$ , due to the definition of  $\nu$ .

**Corollary 2.5.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded domain with smooth boundary. Then, for every  $p \geq 2 - 1/(n - 1)$ , the inequality*

$$\text{Cap}(\Omega) \leq \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|} \left( \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^p d\sigma \right)^{1/p} \quad (2.7)$$

holds true, where  $H$  is the mean curvature of  $\partial\Omega$ . Moreover, the equality is fulfilled for some  $p \geq 2 - 1/(n - 1)$  if and only if  $\Omega$  is a round ball. Finally, letting  $p \rightarrow +\infty$  in the previous inequality, one has that

$$\text{Cap}(\Omega) \leq \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|} \max_{\partial\Omega} \left| \frac{H}{n-1} \right|. \quad (2.8)$$

Moreover, the equality is fulfilled if and only if  $\Omega$  is a round ball.

So far we have used the local feature of the monotonicity, namely the fact that  $U'_p \geq 0$ , to deduce a first group of corollaries of Theorem 1.1. To state further consequences of the main theorem, we now exploit the global feature of the monotonicity, comparing our quantities on different level sets of the function  $u$ . By keeping fixed one of these level sets and letting the other become larger and larger, for every  $t \in (0, 1]$  and  $p \geq 2 - 1/(n - 1)$  we have that

$$U_p(t) \geq \lim_{\tau \rightarrow 0^+} U_p(\tau) = [\text{Cap}(\Omega)]^p (n-2)^p |\mathbb{S}^{n-1}|. \quad (2.9)$$

The latter estimate follows immediately from (1.4) and Theorem 1.1-(ii). Combining this fact with (2.4) and using the definition of  $U_p$ , we have obtained the following chain of sharp inequalities

$$|\mathbb{S}^{n-1}|^{\frac{1}{p}} \left[ \frac{\text{Cap}(\Omega)}{t} \right]^{1 - \frac{(p-1)(n-1)}{p(n-2)}} \leq \left\| \frac{D(\log u)}{n-2} \right\|_{L^p(\{u=t\})} \leq \left\| \frac{H}{n-1} \right\|_{L^p(\{u=t\})},$$

where the equality is fulfilled if and only if  $\Omega$  is a ball and  $u$  is rotationally symmetric. Setting  $t = 1$  in the above formula, we deduce a more geometric statement, involving only the mean curvature of  $\partial\Omega$  and the capacity of  $\Omega$ .

**Theorem 2.6.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded domain with smooth boundary. Then, for every  $p \geq 2 - 1/(n - 1)$ , the inequality*

$$\left( \frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|} \right)^{1/p} \leq [\text{Cap}(\Omega)]^{\frac{p-(n-1)}{p(n-2)}} \left( \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^p d\sigma \right)^{1/p} \quad (2.10)$$

holds true, where  $H$  is the mean curvature of  $\partial\Omega$ . Moreover, the equality is fulfilled for some  $p \geq 2 - 1/(n-1)$  if and only if  $\Omega$  is a round ball. Letting  $p \rightarrow +\infty$  in the previous inequality, one has that

$$1 \leq [\text{Cap}(\Omega)]^{\frac{1}{n-2}} \max_{\partial\Omega} \left| \frac{H}{n-1} \right|. \quad (2.11)$$

Note that every  $n$ -dimensional ball satisfies the equality case in (2.11), and thus the inequality is sharp. At least at a first glance, it is not clear if the converse is also true. However, we conjecture that – at least under our assumptions – this could be the case. This would exclude the existence of non spherically symmetric domains satisfying the equality in (2.11). Finally, combining (2.11) with (2.8) we deduce the following upper and lower bound for the capacity in terms of the maximum of the mean curvature.

$$\left( \frac{(|\mathbb{S}^{n-1}|)^{1/(n-1)}}{|\partial\Omega|} \right)^{n-2} \leq \frac{\text{Cap}(\Omega)}{(|\partial\Omega|)^{\frac{n-2}{n-1}}} \leq \frac{\max_{\partial\Omega} \left| \frac{H}{n-1} \right|}{(|\mathbb{S}^{n-1}|)^{1/(n-1)}} \quad (2.12)$$

A stronger statement, namely the double inequality (2.14) below, will be deduced in few lines. However, we take this occasion to spend some words about the quantity

$$\mathcal{E}(\Omega) = \frac{\text{Cap}(\Omega)}{(|\partial\Omega|)^{\frac{n-2}{n-1}}},$$

given by the ratio between the capacity of  $\Omega$  and the so called Russel capacity or surface radius of  $\Omega$ . Such a quantity has been deeply analyzed by several authors and we refer the interested reader to [14] and to the references therein for a complete overview on the subject. Here, we just recall that in dimension  $n = 3$  a long standing conjecture of Pólya and Szegő asserts that the infimum of  $\Omega \mapsto \mathcal{E}(\Omega)$  among the bounded convex subsets of  $\mathbb{R}^3$  with positive surface measure is given  $(2\sqrt{2})/\pi$  and it is attained if and only if  $\Omega$  is a two dimensional disk.

To obtain further consequences of Theorem 2.6, we observe that, setting  $p = n - 1$  in (2.10), the factor involving the capacity of  $\Omega$  becomes 1 and we can deduce, as a purely geometric consequence of our theory, the well known Willmore inequality, together with the corresponding rigidity statement (see [9, Theorem 3], [10], and also [26]). This is the content of the following corollary.

**Corollary 2.7** (Willmore inequality). *Let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded domain with smooth boundary. Then, the inequality*

$$|\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \quad (2.13)$$

holds true, where  $H$  is the mean curvature of  $\partial\Omega$ . Moreover, the equality is fulfilled if and only if  $\Omega$  is a round ball.

To conclude this section, we notice that for  $p \neq n - 1$  it is possible to use inequality (2.10) in order to obtain upper and lower bounds for the capacity, depending if  $p$  is respectively smaller or bigger than the threshold value  $(n - 1)$ .

**Corollary 2.8.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded domain with smooth boundary. Then, for every*

$$\left( 2 - \frac{1}{n-1} \right) \leq p < (n-1) < q < +\infty$$

*the inequalities*

$$\left[ \frac{(|\mathbb{S}^{n-1}|)^{1/(n-1)}}{|\partial\Omega|} \right]^{\frac{q(n-2)}{q-(n-1)}} \leq \frac{\text{Cap}(\Omega)}{(|\partial\Omega|)^{\frac{n-2}{n-1}}} \leq \left[ \frac{(f_{\partial\Omega} \left| \frac{H}{n-1} \right|^p d\sigma)^{1/p}}{(|\mathbb{S}^{n-1}|)^{1/(n-1)}} \right]^{\frac{p(n-2)}{(n-1)-p}} \quad (2.14)$$

hold true, where  $H$  is the mean curvature of  $\partial\Omega$ . Moreover, the equality is fulfilled for some  $p$  or  $q$  in the above range if and only if  $\Omega$  is a round ball.

It would be interesting to compare the above result to the ones obtained in the very nice paper [27] via the analysis of the  $p$ -capacity.



### 3. A CONFORMALLY EQUIVALENT FORMULATION OF THE PROBLEM

**3.1. A conformal change of metric.** Proceeding as in [2, Section 2], we perform a conformal change of the Euclidean metric to obtain an equivalent formulation of problem (1.1). Consider a solution  $u$  to problem (1.1) and note that  $0 < u < 1$ , by the maximum principle. To set up the notation, we let  $M$  be  $\mathbb{R}^n \setminus \Omega$ , denote by  $g_{\mathbb{R}^n}$  the flat Euclidean metric of  $\mathbb{R}^n$ , and consider the conformally equivalent metric given by

$$g = u^{\frac{2}{n-2}} g_{\mathbb{R}^n}. \quad (3.1)$$

To reformulate our problem it is also convenient to set

$$\varphi = -\log u, \quad (3.2)$$

so that the metric  $g$  can be equivalently written as  $g = e^{-\frac{2\varphi}{n-2}} g_{\mathbb{R}^n}$ . In what follows we denote by  $\langle \cdot | \cdot \rangle$  and  $\langle \cdot | \cdot \rangle_g$  and by  $D$  and  $\nabla$  the scalar products and the covariant derivatives of the metrics  $g_{\mathbb{R}^n}$  and  $g$ , respectively. The symbols  $D^2$ ,  $\nabla^2$  and  $\Delta$ ,  $\Delta_g$  stand for the corresponding Hessian and Laplacian operators. The same computation as in [2, Section 2] show that problem (1.1) is equivalent to

$$\left\{ \begin{array}{ll} \Delta_g \varphi = 0 & \text{in } M, \\ \text{Ric}_g - \nabla^2 \varphi + \frac{d\varphi \otimes d\varphi}{n-2} = \frac{|\nabla \varphi|_g^2}{n-2} g & \text{in } M, \\ \varphi = 0 & \text{on } \partial M, \\ \varphi(x) \rightarrow +\infty & \text{as } x \rightarrow \infty. \end{array} \right. \quad (3.3)$$

Notice that the second equation corresponds to the equation  $\text{Ric}_{g_{\mathbb{R}^n}} = 0$ , which is implicit in the fact that the background metric in problem (1.1) is the flat one. Before proceeding, it is worth observing that the following identities hold true

$$\left( \frac{n-2}{n-1} \right) R_g = |\nabla \varphi|_g^2 = \left( \frac{|Du|}{u^{\frac{n-1}{n-2}}} \right)^2, \quad (3.4)$$

where the first equality is deduced by taking the trace of the second equation in (3.3). This formula says that up to a multiplicative constant, the scalar curvature of the conformally related metric  $g$  coincides with the  $P$ -function  $x \mapsto u^{-2(n-1)/(n-2)} |Du|^2$ . For future convenience, we give the following definition.

**Definition 1.** We say that a solution  $(M, g, \varphi)$  to problem (3.3) has bounded geometry if

$$\sup_M |\nabla \varphi|_g + \sup_M |\nabla^2 \varphi|_g + \sup_{s \geq s_0} \int_{\{\varphi=s\}} d\sigma_g < \infty, \quad (3.5)$$

for some  $s_0 > 0$ .

Essentially, the rest of the paper will be devoted to derive some integral identities for solutions to (3.3) with *bounded geometry*, that imply in turn a conformal version (see Theorem 3.2 below) of the Monotonicity-Rigidity Theorem 1.1. To obtain the original statements, it is then sufficient to observe that if a solution  $(M, g, \varphi)$  to (3.3) comes from a solution of (1.1) via (3.1) and (3.2), then it has necessarily bounded geometry. This is the content of the following lemma.

**Lemma 3.1.** Let  $M = \mathbb{R}^n \setminus \Omega$ , let  $u$  be a solution to problem (1.1) and let  $g$  and  $\varphi$  be given by  $g_{\mathbb{R}^n}$  and  $u$  through (3.1) and (3.2). Then  $(M, g, \varphi)$  is a solution to (3.3) with bounded geometry.

*Proof.* It is enough using the asymptotic estimates (1.2) and arguing as in [2, proof of Proposition 3.3].  $\square$

**3.2. The geometry of the level sets of  $\varphi$ .** For the forthcoming analysis it is important to study the geometry of the level sets of  $\varphi$ , which coincide with the level sets of  $u$  by definition. To this end, we fix on  $M$  the  $g_{\mathbb{R}^n}$ -unit vector field  $\nu = -Du/|Du| = D\varphi/|D\varphi|$  and the  $g$ -unit vector field  $\nu_g = -\nabla u/|\nabla u|_g = \nabla \varphi/|\nabla \varphi|_g$ . Consequently, the second fundamental forms of the regular level sets of  $u$  or  $\varphi$  with respect to the flat ambient metric and the conformally-related ambient metric  $g$ , are given by

$$\begin{aligned} h(X, Y) &= -\frac{D^2 u(X, Y)}{|Du|} = -\frac{D^2 \varphi(X, Y)}{|D\varphi|}, \\ h_g(X, Y) &= -\frac{\nabla^2 u(X, Y)}{|\nabla u|_g} = -\frac{\nabla^2 \varphi(X, Y)}{|\nabla \varphi|_g}, \end{aligned}$$

respectively, where  $X$  and  $Y$  are vector fields tangent to the level sets. Taking the traces of the above expressions with respect to the induced metrics and using the fact that  $u$  is harmonic and  $\varphi$  is  $g$ -harmonic, we obtain the following expressions for the mean curvatures in the two ambients

$$H = \frac{D^2u(Du, Du)}{|Du|^3}, \quad H_g = -\frac{\nabla^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|_g^3}. \quad (3.6)$$

By a direct computation one can show that the second fundamental forms and the mean curvatures are related by the following formulæ

$$h_g(X, Y) = u^{\frac{1}{n-2}} \left[ h(X, Y) - \left( \frac{1}{n-2} \right) \frac{|Du|}{u} \langle X|Y \rangle \right], \quad (3.7)$$

$$H_g = u^{-\frac{1}{n-2}} \left[ H - \left( \frac{n-1}{n-2} \right) \frac{|Du|}{u} \right], \quad (3.8)$$

where, as before,  $X$  and  $Y$  are vector fields tangent to the level sets. For the sake of completeness, we also report the reversed formulæ

$$h(X, Y) = e^{\frac{\varphi}{n-2}} \left[ h_g(X, Y) + \left( \frac{1}{n-2} \right) |\nabla\varphi|_g \langle X|Y \rangle_g \right],$$

$$H = e^{-\frac{\varphi}{n-2}} \left[ H_g + \left( \frac{n-1}{n-2} \right) |\nabla\varphi|_g \right].$$

Concerning the nonregular level sets of  $\varphi$ , we just observe that, by the same arguments as in Remark 1, the second fundamental form and mean curvature also make sense  $\mathcal{H}^{n-1}$ -almost everywhere on a singular level set  $\{\varphi = s_0\}$ , namely on the relatively open set  $\{\varphi = s_0\} \setminus \text{Crit}(\varphi)$ .

**3.3. A conformal version of the Monotonicity-Rigidity Theorem.** In this subsection, we deal with the conformal analog of the functions

$$t \mapsto U_p(t) = \left[ \frac{\text{Cap}(\Omega)}{t} \right]^{(p-1)\left(\frac{n-1}{n-2}\right)} \int_{\{u=t\}} |Du|^p d\sigma,$$

introduced in (1.3). Given a solution  $(M, g, \varphi)$  to problem (3.3) with bounded geometry and given  $p \geq 0$ , we define the function  $\Phi_p : [0, +\infty) \rightarrow \mathbb{R}$  as

$$s \mapsto \Phi_p(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_g^p d\sigma_g. \quad (3.9)$$

**Remark 4.** By the same considerations as in Remark 1, we observe that  $\Phi_p(s)$  and the integral in formula (3.12) below are well defined on every level set  $\{\varphi = s\}$ , due to the fact that  $\varphi$  is harmonic and proper, and using the results of [8, 17, 18].

We say that  $\Phi_p$  is the conformal analog of  $U_p$  because these functions and their derivatives, when defined, are related as follows

$$U_p(t) = [\text{Cap}(\Omega)]^{(p-1)\left(\frac{n-1}{n-2}\right)} \Phi_p(-\log t), \quad (3.10)$$

$$-t U_p'(t) = [\text{Cap}(\Omega)]^{(p-1)\left(\frac{n-1}{n-2}\right)} \Phi_p'(-\log t). \quad (3.11)$$

Note that, when  $p = 0$ , the function

$$\Phi_0(s) = \int_{\{\varphi=s\}} d\sigma_g = |\{\varphi = s\}|_g,$$

coincides with the hypersurface area functional  $|\{\varphi = s\}|_g$  for the level sets of  $\varphi$  inside the ambient manifold  $(M, g)$ . For  $p = 1$ , it follows from  $\Delta_g\varphi = 0$  and the Divergence Theorem that the function

$$\Phi_1(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_g d\sigma_g$$

is constant. For the sake of completeness, we also observe that (1.4) and (3.10) imply that

$$\lim_{s \rightarrow +\infty} \Phi_p(s) = [\text{Cap}(\Omega)]^{\frac{n-1-p}{n-2}} (n-2)^p |\mathbb{S}^{n-1}|.$$



Using the identities (3.10)–(3.11), the Monotonicity-Rigidity Theorem 1.1 can be rephrased in terms of the functions  $\Phi_p$ 's as follows.

**Theorem 3.2** (Monotonicity-Rigidity Theorem – Conformal Version). *Let  $(M, g, \varphi)$  be a solution to problem (3.3) with bounded geometry in the sense of Definition 1 and suppose that  $\partial M$  is a regular level set of  $\varphi$ . Let  $\Phi_p : [0, +\infty) \rightarrow \mathbb{R}$  be the function defined in (3.9). Then, the following properties hold true.*

- (i) *For every  $p \geq 1$ , the function  $\Phi_p$  is continuous.*
- (ii) *For every  $p \geq 2 - 1/(n - 1)$ , the function  $\Phi_p$  is differentiable and the derivative satisfies, for every  $s \geq 0$ ,*

$$\Phi_p'(s) = -(p - 1) \int_{\{\varphi=s\}} |\nabla \varphi|_g^{p-1} H_g d\sigma_g \leq 0, \quad (3.12)$$

where  $H_g$  is the mean curvature of the level set  $\{\varphi = s\}$ . Moreover, if  $\Phi_p'(s) = 0$  for some  $s \geq 0$ , then  $(M, g, \varphi)$  is isometric to one half round cylinder with totally geodesic boundary.

Note that in view of (1.8), (3.8), and of (3.11), formula (3.12) is equivalent to formula (1.7) in Theorem 1.1. Since it is now clear that Theorem 1.1 is completely equivalent to Theorem 3.2, the rest of the paper is devoted to the proof of Theorem 3.2.

**Remark 5.** *An argument similar to that of Remark 1 easily shows that for  $p \geq 2$  the integral in (3.12) is finite also when  $s$  is a critical value of  $\varphi$ . In the range  $2 - 1/(n - 1) \leq p < 2$ , the integrand term  $|\nabla \varphi|_g^{p-1} H_g$  is unbounded around the critical points of  $\varphi$ . Nevertheless, the integral still converges, as it can be seen from a perusal of the proof of Proposition 4.2, through which Theorem 3.2 is proved.*

#### 4. INTEGRAL IDENTITIES

In this section, we derive some integral identities that will be used to analyze the properties of the functions  $\Phi_p$ 's introduced in (3.9). Because of the prominent role of  $\nabla \varphi$  in the definition of the  $\Phi_p$ 's, it is important to treat carefully our identities in the region around  $\text{Crit}(\varphi)$ . This will be done using the results gathered in the Appendix.

The equation  $\Delta_g \varphi = 0$ , combined with the bounds (3.5) imposed by the bounded geometry, yields a first identity, which will prove to be a useful tool for the study of the continuity of the  $\Phi_p$ 's.

**Proposition 4.1** (First integral identity). *Let  $(M, g, \varphi)$  be a solution to problem (3.3) with bounded geometry in the sense of Definition 1. Then, for every  $p \geq 1$  and every  $s \geq 0$ , we have*

$$\int_{\{\varphi=s\}} \frac{|\nabla \varphi|_g^p}{e^s} d\sigma_g = \int_{\{\varphi>s\}} \frac{|\nabla \varphi|_g^{p-3} \left( |\nabla \varphi|_g^4 - (p - 1) \nabla^2 \varphi(\nabla \varphi, \nabla \varphi) \right)}{e^\varphi} d\mu_g. \quad (4.1)$$

*Proof.* To simplify the notation, we drop the subscript  $g$  throughout the proof. We consider the vector field

$$X := \frac{|\nabla \varphi|^{p-1} \nabla \varphi}{e^\varphi},$$

and compute, wherever  $|\nabla \varphi| > 0$ ,

$$\text{div} X = \frac{|\nabla \varphi|^{p-3} \left( (p - 1) \nabla^2 \varphi(\nabla \varphi, \nabla \varphi) - |\nabla \varphi|^4 \right)}{e^\varphi}, \quad (4.2)$$

using the fact that  $\varphi$  is harmonic. Note that  $\text{div} X$  is bounded all over  $M$ , being  $p \geq 1$ . Now, choosing  $S > s \geq 0$ , define  $E := \{s < \varphi < S\}$  and recall from [25] that there exists a finite number of critical values in between  $s$  and  $S$ . For notational simplicity, let us now assume that there is only one critical value  $\bar{s}$  of  $\varphi$  in  $(s, S)$ . This assumption is not restrictive, since to treat the general case it is enough adapting the notation of the following argument. Now, let  $\{U_\varepsilon(\bar{s})\}$  be a (nonincreasing) family of *stratified tubular neighbourhoods* of  $\{\varphi = \bar{s}\} \cap \{\nabla \varphi = 0\}$ , given by Proposition A.3 and Corollary A.4. Note that the values  $s$  and  $S$  can be singular as well. In this case, we denote by  $\{U_\varepsilon(s)\}$  and  $\{U_\varepsilon(S)\}$  the corresponding families of stratified tubular neighbourhoods, and denote by  $U_\varepsilon$  the union of  $U_\varepsilon(s)$ ,  $U_\varepsilon(\bar{s})$ , and  $U_\varepsilon(S)$ . We refer the reader to

Figure 4 for a sketch of the set  $U_\varepsilon$ . By the Dominated Convergence Theorem, we have that

$$\begin{aligned}
& \int_E \frac{|\nabla\varphi|^{p-3} \left( (p-1) \nabla^2\varphi(\nabla\varphi, \nabla\varphi) - |\nabla\varphi|^4 \right)}{e^\varphi} d\mu \\
&= \lim_{\varepsilon \downarrow 0} \int_{E \setminus U_\varepsilon} \operatorname{div} X d\mu = \lim_{\varepsilon \downarrow 0} \int_{\partial(E \setminus U_\varepsilon)} \langle X | n \rangle d\sigma \\
&= \lim_{\varepsilon \downarrow 0} \left\{ \int_{\{\varphi=s\} \setminus U_\varepsilon(s)} \langle X | n \rangle d\sigma + \int_{\partial U_\varepsilon(s) \cap E} \langle X | n \rangle d\sigma + \int_{\partial U_\varepsilon(s)} \langle X | n \rangle d\sigma + \int_{\{\varphi=S\} \setminus U_\varepsilon(S)} \langle X | n \rangle d\sigma + \int_{\partial U_\varepsilon(S) \cap E} \langle X | n \rangle d\sigma \right\}, \tag{4.3}
\end{aligned}$$

where  $n$  is the outer  $g$ -unit normal of the set  $E \setminus U_\varepsilon$  at its boundary. Notice that to obtain the second equality we have applied the standard Divergence Theorem for smooth vector fields. Observe now that since  $X$  is bounded, from the first of estimates (A-9) in Corollary A.4 we get

$$\left| \int_{\partial U_\varepsilon(s) \cap E} \langle X | n \rangle d\sigma + \int_{\partial U_\varepsilon(s)} \langle X | n \rangle d\sigma + \int_{\partial U_\varepsilon(S) \cap E} \langle X | n \rangle d\sigma \right| \leq \int_{\partial U_\varepsilon} |\langle X | n \rangle| d\sigma \leq Cd\varepsilon. \tag{4.4}$$

Note also that, again by the Dominated Convergence Theorem, we have that

$$\lim_{\varepsilon \downarrow 0} \left\{ \int_{\{\varphi=s\} \setminus U_\varepsilon(s)} \langle X | n \rangle d\sigma + \int_{\{\varphi=S\} \setminus U_\varepsilon(S)} \langle X | n \rangle d\sigma \right\} = \int_{\{\varphi=s\}} \langle X | n \rangle d\sigma + \int_{\{\varphi=S\}} \langle X | n \rangle d\sigma.$$

From this identity, and from (4.3)–(4.4), we obtain

$$\int_E \frac{|\nabla\varphi|^{p-3} \left( (p-1) \nabla^2\varphi(\nabla\varphi, \nabla\varphi) - |\nabla\varphi|^4 \right)}{e^\varphi} d\mu = \int_{\{\varphi=s\}} \langle X | n \rangle d\sigma + \int_{\{\varphi=S\}} \langle X | n \rangle d\sigma. \tag{4.5}$$

Finally, note that  $n = -\nabla\varphi/|\nabla\varphi|$  on  $\{\varphi = s\}$  and  $n = \nabla\varphi/|\nabla\varphi|$  on  $\{\varphi = S\}$ , and that, thanks to (3.5), we have

$$\lim_{S \rightarrow +\infty} \int_{\{\varphi=S\}} \frac{|\nabla\varphi|^p}{e^\varphi} d\sigma = 0.$$

Hence, identity (4.1) can be directly derived taking the limit as  $S \rightarrow +\infty$  of (4.5).  $\square$

To obtain the second relevant integral identity, a few preliminary observations are in order. First, consider the Bochner formula

$$\frac{1}{2} \Delta_g |\nabla\varphi|_g^2 = |\nabla^2\varphi|_g^2 + \operatorname{Ric}_g(\nabla\varphi, \nabla\varphi) + \langle \nabla \Delta_g \varphi | \nabla\varphi \rangle_g,$$

and notice that it reduces to

$$\Delta_g |\nabla\varphi|_g^2 - \langle \nabla |\nabla\varphi|_g^2 | \nabla\varphi \rangle_g = 2 |\nabla^2\varphi|_g^2, \tag{4.6}$$

when using the first two equations in (3.3). Also, observe that wherever  $|\nabla\varphi|_g > 0$  it holds

$$\begin{aligned}
|\nabla |\nabla\varphi|_g^{p-1}| &= \left( \frac{p-1}{2} \right) |\nabla\varphi|_g^{p-3} |\nabla |\nabla\varphi|_g^2|, \\
\Delta_g |\nabla\varphi|_g^{p-1} &= \left( \frac{p-1}{2} \right) |\nabla\varphi|_g^{p-3} \Delta_g |\nabla\varphi|_g^2 + (p-1)(p-3) |\nabla\varphi|_g^{p-3} |\nabla |\nabla\varphi|_g^2|_g^2,
\end{aligned}$$

and identity (4.6) can be extended to

$$\Delta_g |\nabla\varphi|_g^{p-1} - \langle \nabla |\nabla\varphi|_g^{p-1} | \nabla\varphi \rangle_g = (p-1) |\nabla\varphi|_g^{p-3} \left( |\nabla^2\varphi|_g^2 + (p-3) |\nabla |\nabla\varphi|_g^2|_g^2 \right). \tag{4.7}$$

This formula coincides with (4.6) when  $p = 3$ . Integrating it with respect to the weighted measure  $(1/e^\varphi) d\mu_g$  allows us to prove our second integral identity, which is the content of the following proposition and will be used later on to derive the monotonicity properties of the  $\Phi_p$ 's.

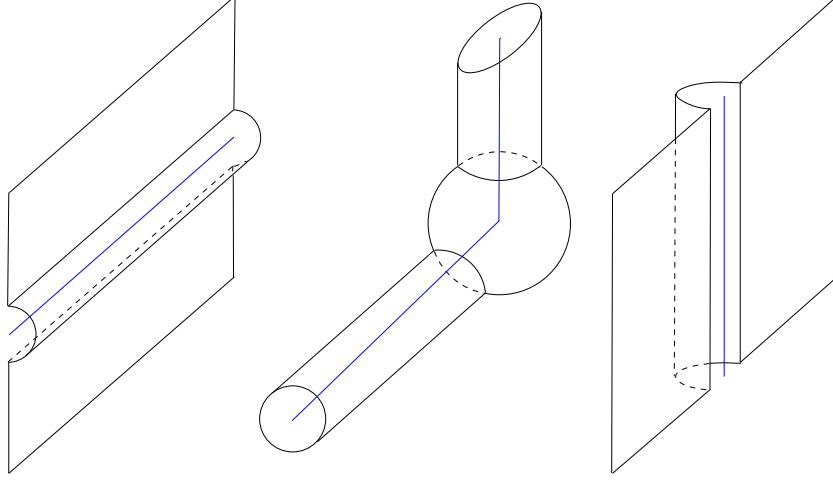


FIGURE 1. A possible appearance of (a section of)  $\text{Crit}(\varphi) \cap \{s < \varphi < S\}$  is given by the blue lines:  $\text{Crit}(\varphi) \cap \{\varphi = s\}$  on the left,  $\text{Crit}(\varphi) \cap \{\varphi = \bar{s}\}$  in the middle,  $\text{Crit}(\varphi) \cap \{\varphi = S\}$  on the right. The black lines emphasize the boundary of  $\{s < \varphi < S\} \setminus U_\varepsilon$ , where  $U_\varepsilon$  is the union of the stratified tubular neighbourhoods  $U_\varepsilon(s)$ ,  $U_\varepsilon(\bar{s})$ , and  $U_\varepsilon(S)$ .

**Proposition 4.2** (Second integral identity). *Let  $(M, g, \varphi)$  be a solution to problem (3.3) with bounded geometry in the sense of Definition 1. Then, for every  $p \geq 2 - 1/(n - 1)$  and every  $s \geq 0$ , we have*

$$\int_{\{\varphi=s\}} \frac{|\nabla \varphi|_g^{p-1} H_g}{e^s} d\sigma_g = \int_{\{\varphi>s\}} \frac{|\nabla \varphi|_g^{p-3} \left( |\nabla^2 \varphi|_g^2 + (p-3) |\nabla |\nabla \varphi|_g|^2 \right)}{e^\varphi} d\mu_g \geq 0. \quad (4.8)$$

Moreover, if there exists  $s_0 \geq 0$  such that

$$\int_{\{\varphi=s_0\}} |\nabla \varphi|_g^{p_0-1} H_g d\sigma_g \leq 0, \quad (4.9)$$

for some  $p_0 \geq 2 - 1/(n - 1)$ , then the manifold  $(\{\varphi \geq s_0\}, g)$  is isometric to  $([s_0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + g|_{\{\varphi=s_0\}})$ , where  $\rho$  is the  $g$ -distance to  $\{\varphi = s_0\}$ , and  $\varphi$  is an affine function of  $\rho$ .

**Remark 6.** Observe that if a solution  $(M, g, \varphi)$  to problem (3.3) is analytic - as in the case when  $(M, g, \varphi)$  comes from a solution  $u$  to problem (1.1) through (3.1) and (3.2) - the conclusion of the rigidity statement in Proposition 4.2 are stronger. More precisely, the isometry between  $(\{\varphi \geq s_0\}, g)$  and  $([s_0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + g|_{\{\varphi=s_0\}})$  improves to an isometry between the whole manifold  $(M, g)$  and  $([0, +\infty) \times \{\varphi = 0\}, d\rho \otimes d\rho + g|_{\{\varphi=0\}})$ .

*Proof of Proposition 4.2.* As in the proof of Proposition 4.1, we drop the subscript  $g$  for notational simplicity. For every  $p \geq 2 - 1/(n - 1)$ , we consider the vector field

$$Y := \frac{\nabla |\nabla \varphi|^{p-1}}{e^\varphi}. \quad (4.10)$$

Observe that, due to formula (4.7), we have

$$\begin{aligned} \text{div} Y &= \frac{\Delta |\nabla \varphi|^{p-1} - \langle \nabla |\nabla \varphi|^{p-1} | \nabla \varphi \rangle}{e^\varphi} \\ &= (p-1) \frac{|\nabla \varphi|^{p-3} \left( |\nabla^2 \varphi|^2 + (p-3) |\nabla |\nabla \varphi||^2 \right)}{e^\varphi}, \end{aligned} \quad (4.11)$$

wherever  $|\nabla \varphi| > 0$ . Now, chosen  $S > s \geq 0$ , we are going to integrate identity (4.11) on  $E := \{s < \varphi < S\}$ . As in the proof of Proposition 4.1, we suppose for notational simplicity that there is only one critical value  $\bar{s}$  of  $\varphi$  in  $(s, S)$ . Before proceeding, a comment regarding the sign of the right-hand side of (4.11) is in order. Since  $p$  is bounded below by  $2 - 1/(n - 1)$ , we have that  $(p - 3) \geq -n/(n - 1)$  and in turn the right-hand

side of (4.11) is a.e. nonnegative in  $M$ , due to the refined Kato inequality for harmonic functions

$$|\nabla^2 \varphi|^2 \geq \frac{n}{n-1} |\nabla |\nabla \varphi||^2,$$

which holds whenever  $|\nabla \varphi| > 0$ . Now, let  $\{U_\varepsilon(\bar{s})\}$  be a nonincreasing family of *stratified tubular neighbourhoods* of  $\{\varphi = \bar{s}\} \cap \{\nabla \varphi = 0\}$ , according to Proposition A.3. If the values  $s$  and  $S$  are singular as well, we denote by  $\{U_\varepsilon(s)\}$  and  $\{U_\varepsilon(S)\}$  the corresponding families of stratified tubular neighbourhoods, and set  $U_\varepsilon := U_\varepsilon(s) \cup U_\varepsilon(\bar{s}) \cup U_\varepsilon(S)$ . Using the Monotone Convergence Theorem first and then the Divergence Theorem, we deduce that

$$\begin{aligned} & (p-1) \int_E \frac{|\nabla \varphi|^{p-3} \left( |\nabla^2 \varphi|^2 + (p-3) |\nabla |\nabla \varphi||^2 \right)}{e^\varphi} d\mu \\ &= \lim_{\varepsilon \downarrow 0} \int_{E \setminus U_\varepsilon} \operatorname{div} Y d\mu = \lim_{\varepsilon \downarrow 0} \int_{\partial(E \setminus U_\varepsilon)} \langle Y | n \rangle d\sigma \\ &= \lim_{\varepsilon \downarrow 0} \left\{ \int_{\{\varphi=s\} \setminus U_\varepsilon(s)} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(s) \cap E} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(\bar{s})} \langle Y | n \rangle d\sigma + \int_{\{\varphi=S\} \setminus U_\varepsilon(S)} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(S) \cap E} \langle Y | n \rangle d\sigma \right\}. \end{aligned} \quad (4.12)$$

We now want to show that

$$\lim_{\varepsilon \downarrow 0} \left\{ \int_{\partial U_\varepsilon(s) \cap E} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(\bar{s})} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(S) \cap E} \langle Y | n \rangle d\sigma \right\} = 0. \quad (4.13)$$

To do this, we distinguish the case  $p \geq 2$  from the case  $2 - 1/(n-1) \leq p < 2$ . In the range  $p \geq 2$ , we note that the vector field  $Y$  is bounded, since

$$|Y| \leq (p-1) \frac{|\nabla \varphi|^{p-2} |\nabla^2 \varphi|}{e^\varphi} \leq C < +\infty, \quad (4.14)$$

due to the Kato inequality and to (3.5). Hence, it is enough using the fact that  $\mathcal{H}^{n-1}(\partial U_\varepsilon) \downarrow 0$  as  $\varepsilon \downarrow 0$  to accomplish the program. More precisely, using the first of estimates (A-9) in Corollary A.4 we obtain that

$$\left| \int_{\partial U_\varepsilon(s) \cap E} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(\bar{s})} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(S) \cap E} \langle Y | n \rangle d\sigma \right| \leq \int_{\partial U_\varepsilon} |\langle Y | n \rangle| d\sigma \leq C d\varepsilon. \quad (4.15)$$

In the range  $2 - 1/(n-1) \leq p < 2$ , the quantity  $\langle Y | n \rangle$  is no longer bounded on  $\partial U_\varepsilon$  and an estimate of  $\mathcal{H}^{n-1}(\partial U_\varepsilon)$  has to be coupled with an information on how big the quantity  $\langle Y | n \rangle$  is in terms of  $\varepsilon$ . This is given by Proposition A.3. In turn, this implies the second estimate of (A-9) in Corollary A.4, which yields

$$\begin{aligned} & \left| \int_{\partial U_\varepsilon(s) \cap E} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(\bar{s})} \langle Y | n \rangle d\sigma + \int_{\partial U_\varepsilon(S) \cap E} \langle Y | n \rangle d\sigma \right| \\ & \leq \int_{\partial U_\varepsilon(s)} |\langle Y | n \rangle| d\sigma + \int_{\partial U_\varepsilon(\bar{s})} |\langle Y | n \rangle| d\sigma + \int_{\partial U_\varepsilon(S)} |\langle Y | n \rangle| d\sigma \leq [c(s) + c(\bar{s}) + c(S)] \varepsilon^{\frac{p-1}{2}}. \end{aligned} \quad (4.16)$$

Both cases (4.15) and (4.16) imply (4.13). Before proceeding, consider for a moment the case where both  $s$  and  $S$  are regular values of  $\varphi$ . In this case, we have from the previous analysis that

$$\begin{aligned} (p-1) \int_E \frac{|\nabla\varphi|^{p-3} \left( |\nabla^2\varphi|^2 + (p-3) |\nabla|\nabla\varphi||^2 \right)}{e^\varphi} d\mu \\ = \int_{\{\varphi=s\}} \langle Y | n \rangle d\sigma + \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(\bar{s})} \langle Y | n \rangle d\sigma + \int_{\{\varphi=S\}} \langle Y | n \rangle d\sigma \\ = \int_{\{\varphi=s\}} \langle Y | n \rangle d\sigma + \int_{\{\varphi=S\}} \langle Y | n \rangle d\sigma, \end{aligned}$$

and from the finiteness of the right-hand side in the previous identity we deduce that the function

$$x \mapsto |\nabla\varphi|^{p-3} \left( |\nabla^2\varphi|^2 + (p-3) |\nabla|\nabla\varphi||^2 \right)$$

has finite integral on  $\{s < \varphi < S\}$ . In turn, we deduce that

$$\int_E \frac{|\nabla\varphi|^{p-3} \left( |\nabla^2\varphi|^2 + (p-3) |\nabla|\nabla\varphi||^2 \right)}{e^\varphi} d\mu < +\infty, \quad (4.17)$$

also in the case where either  $s$  or  $S$  is a singular value of  $\varphi$ , because, by the nonnegativity of the integrand function, the integral on  $\{s < \varphi < S\}$  can be always bounded by the same integral on  $\{\tilde{s} < \varphi < \tilde{S}\}$ , where  $\tilde{s}$  and  $\tilde{S}$  are regular values of  $\varphi$  smaller than  $s$  and bigger than  $S$ , respectively. Now, let's go back to identity (4.12), which in particular implies, coupled with (4.12) and with (4.17), that the limit

$$\lim_{\varepsilon \downarrow 0} \left\{ \int_{\{\varphi=s\} \setminus U_\varepsilon(s)} \langle Y | n \rangle d\sigma + \int_{\{\varphi=S\} \setminus U_\varepsilon(S)} \langle Y | n \rangle d\sigma \right\}$$

exists and is finite. Therefore, using Vitali's Convergence Theorem (see for instance [24, Chapter 6]), we obtain that the previous limit coincides with

$$\int_{\{\varphi=s\}} \langle Y | n \rangle d\sigma + \int_{\{\varphi=S\}} \langle Y | n \rangle d\sigma.$$

This fact, again together with (4.12)–(4.13), finally implies

$$(p-1) \int_E \frac{|\nabla\varphi|^{p-3} \left( |\nabla^2\varphi|^2 + (p-3) |\nabla|\nabla\varphi||^2 \right)}{e^\varphi} d\mu = \int_{\{\varphi=s\}} \langle Y | n \rangle d\sigma + \int_{\{\varphi=S\}} \langle Y | n \rangle d\sigma. \quad (4.18)$$

Notice that  $n = -\nabla\varphi/|\nabla\varphi|$  on  $\{\varphi = s\}$  and  $n = \nabla\varphi/|\nabla\varphi|$  on  $\{\varphi = S\}$ , and that the second formula in (3.6) gives

$$\langle \nabla|\nabla\varphi|^{p-1} \nabla\varphi \rangle = (p-1) |\nabla\varphi|^{p-3} \nabla^2\varphi(\nabla\varphi, \nabla\varphi) = -(p-1) |\nabla\varphi|^p H.$$

Hence, from (4.18) we get

$$\int_E \frac{|\nabla\varphi|^{p-3} \left( |\nabla^2\varphi|^2 + (p-3) |\nabla|\nabla\varphi||^2 \right)}{e^\varphi} d\mu = \int_{\{\varphi=s\}} \frac{|\nabla\varphi|^{p-1} H}{e^s} d\sigma - \int_{\{\varphi=S\}} \frac{|\nabla\varphi|^{p-1} H}{e^S} d\sigma. \quad (4.19)$$

In order to obtain identity (4.8) it is now sufficient to show that the last term on the right hand side of (4.19) tends to zero as  $S \rightarrow +\infty$ . In fact, we have that  $|\nabla\varphi|^{p-1} H \leq |\nabla\varphi|^{p-2} |\nabla^2\varphi|$  and in turn that  $\int_{\{\varphi=S\}} |\nabla\varphi|^{p-1} H d\sigma$  is uniformly bounded, in view of (3.5).

To prove the rigidity statement, observe that if (4.9) holds for some  $s_0 \geq 0$  and some  $p_0 \geq 2 - 1/(n-1)$ , then

$$|\nabla^2\varphi|^2 + (p_0 - 3) |\nabla|\nabla\varphi||^2 \equiv 0,$$

in  $\{\varphi \geq s_0\}$ , due to the refined Kato inequality for harmonic functions. Let us now distinguish the case  $p_0 > 2 - 1/(n-1)$  from the case  $p_0 = 2 - 1/(n-1)$ . In the former case it is immediate to conclude that  $|\nabla|\nabla\varphi|| \equiv 0$  in  $\{\varphi \geq s_0\}$ . In turn,  $|\nabla\varphi|$  is constant in that region, and such constant must be positive, due

to the last two conditions in (3.3). Therefore,  $\nabla\varphi$  is a nontrivial parallel vector field and by [2, Theorem 4.1-(i)] we deduce that the Riemannian manifold  $(\{\varphi \geq s_0\}, g)$  is isometric to the manifold  $\{\varphi = s_0\} \times [s_0, +\infty)$  endowed with the product metric  $d\varrho \otimes d\varrho + g|_{\{\varphi=s_0\}}$ , where  $\varrho$  represents the distance to  $\{\varphi = s_0\}$ . Moreover, from the proof of Theorem 4.1-(i) in the mentioned paper, one gets that the function  $\varphi$  can be expressed as an affine function of  $\varrho$  in  $\{\varphi \geq s_0\}$ , i.e.  $\varphi = s_0 + \varrho |\nabla\varphi|_g$ , where  $|\nabla\varphi|_g$  is a positive constant.

Let us now consider the case  $(p_0 - 3) = -n/(n - 1)$ , which gives

$$|\nabla^2\varphi|^2 = \frac{n}{n-1} |\nabla|\nabla\varphi||^2, \quad (4.20)$$

in  $\{\varphi \geq s_0\}$ . Following the proof of [6, Proposition 5.1], it is possible to deduce that  $|\nabla\varphi|$  is constant along the level sets of  $\varphi$  and thus that the metric  $g$  has a warped product structure in this region, namely

$$g = d\varrho \otimes d\varrho + \eta^2(\varrho) g|_{\{\varphi=s_0\}}, \quad (4.21)$$

for some positive warping function  $\eta = \eta(\varrho)$ . Moreover  $\varphi$  and  $\varrho$  satisfy the relationship

$$\varphi(q) = s_0 + \kappa \int_0^{\varrho(q)} \frac{d\tau}{\eta(\tau)^{n-1}}$$

for every  $q \in \{\varphi \geq s_0\}$  and some  $\kappa \geq 0$ . In particular,  $\varphi$  and  $\varrho$  share the same level sets and, by formula (4.21), these are totally umbilic. In fact one has

$$h_{ij}^{(g)} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \varrho} = \frac{d \log \eta}{d \varrho} g_{ij}.$$

As a consequence, the mean curvature is constant along each level set of  $\varphi$ . Applying formula (4.8), to every level set  $\{\varphi = s\}$  with  $s \geq s_0$  and  $p_0 = 2 - 1/(n - 1)$ , one gets

$$H \int_{\{\varphi=s\}} |\nabla\varphi|^{\frac{n-2}{n-1}} d\sigma = 0,$$

since in virtue of (4.20) the right hand side of (4.8) is always zero. This implies in turn that all the level sets  $\{\varphi = s\}$  with  $s \geq s_0$  are minimal and thus totally geodesic. From  $H \equiv 0$  one can also deduce that  $\langle \nabla|\nabla\varphi|^2 | \nabla\varphi \rangle \equiv 0$  in  $\{\varphi \geq s_0\}$ . Hence,  $|\nabla\varphi|$  is constant in  $\{\varphi \geq s_0\}$ , and thus the conclusion follows arguing as in the case  $p_0 > 2 - 1/(n - 1)$ . □

## 5. PROOF OF THEOREM 3.2

Building on the analysis of the previous section, we are now in the position to prove Theorem 3.2, which in turn implies Theorem 1.1.

**5.1. Continuity.** We claim that under the hypotheses of Theorem 3.2 the function  $\Phi_p$  is continuous, for  $p \geq 1$ . We first observe that since we are assuming that the boundary  $\partial M$  is a regular level set of  $\varphi$ , the function  $s \mapsto \Phi_p(s)$  can be described in terms of an integral depending on the parameter  $s$ , provided  $s \in [0, 2\varepsilon)$ , with  $\varepsilon > 0$  sufficiently small. In this case, the continuous dependence on the parameter  $s$  can be easily checked using standard results from classical differential calculus. Thus, we leave the details to the interest reader and we pass to consider the case where  $s \in (\varepsilon, +\infty)$ . Thanks to Proposition 4.1, one can rewrite expression (3.9) as

$$\Phi_p(s) = e^s \int_{\{\varphi>s\}} \frac{|\nabla\varphi|_g^{p-3} \left( |\nabla\varphi|_g^4 - (p-1) \nabla^2\varphi(\nabla\varphi, \nabla\varphi) \right)}{e^\varphi} d\mu_g. \quad (5.1)$$

It is now convenient to set

$$\mu_g^{(p)}(E) := \int_E \frac{|\nabla\varphi|_g^{p-3} \left( |\nabla\varphi|_g^4 - (p-1) \nabla^2\varphi(\nabla\varphi, \nabla\varphi) \right)}{e^\varphi} d\mu_g, \quad (5.2)$$

for every  $\mu_g$ -measurable set  $E \subseteq \{\varphi > \varepsilon\}$ . It is then clear that for  $p \geq 1$  the measure  $\mu_g^{(p)}$  is absolutely continuous with respect to  $\mu_g$ , since  $|\nabla\varphi|_g^{p-3} \nabla^2\varphi(\nabla\varphi, \nabla\varphi) \leq |\nabla\varphi|_g^{p-1} |\nabla^2\varphi|_g$ . Notice also that, in view

of (5.1), the function  $s \mapsto \Phi_p(s)$  can be interpreted as the repartition function of the measure defined in (5.2), up to the smooth factor  $e^s$ . Thus,  $s \mapsto \Phi_p(s)$  is continuous if and only if the assignment

$$s \longmapsto \mu_g^{(p)}(\{\varphi > s\})$$

is continuous. Thanks to [4, Proposition 2.6], which can be used since  $\mu_g^{(p)}$  is finite due to identity (4.1), proving the continuity of the above assignment is equivalent to checking that  $\mu_g(\{\varphi = s\}) = 0$ , for every  $s > \varepsilon$ . On the other hand, each level set of  $\varphi$  has finite  $\mathcal{H}^{n-1}$ -measure (see e.g. [18]). Hence, each level set of  $\varphi$  is negligible with respect to the full  $n$ -dimensional measure, and in turn negligible with respect to  $\mu_g$ . This proves the continuity of  $\Phi_p$  for  $p \geq 1$ , under the hypotheses of Theorem 3.2.

**5.2. Differentiability.** We now turn our attention to the question of the differentiability of the functions  $s \mapsto \Phi_p(s)$ . As already observed in the previous subsection, we are assuming that the boundary  $\partial M$  is a regular level set of  $\varphi$  so that the function  $s \mapsto \Phi_p(s)$  can be described in term of an integral depending on the parameter  $s$ , provided  $s \in [0, 2\varepsilon]$  with  $\varepsilon > 0$  sufficiently small. Again, the differentiability in the parameter  $s$  can be easily checked in this case, using standard results from classical differential calculus. Leaving the details to the interest reader, we pass to consider the case where  $s \in (\varepsilon, +\infty)$ . Note that we can apply the coarea formula (see [15, Chapter 3]) to expression (5.1), obtaining

$$\begin{aligned} \Phi_p(s) &= e^s \int_{\{\tau > s\}} \int_{\{\varphi = \tau\}} \frac{|\nabla \varphi|_g^p - (p-1) |\nabla \varphi|_g^{p-4} \nabla^2 \varphi(\nabla \varphi, \nabla \varphi)}{e^\varphi} d\sigma_g d\tau \\ &= e^s \int_{\{\tau > s\}} \int_{\{\varphi = \tau\}} \frac{|\nabla \varphi|_g^p + (p-1) |\nabla \varphi|_g^{p-1} H_g}{e^\varphi} d\sigma_g d\tau \\ &= e^s \int_{\{\tau > s\}} \left( \frac{\Phi_p(\tau)}{e^\tau} + (p-1) \int_{\{\varphi = \tau\}} \frac{|\nabla \varphi|_g^{p-1} H_g}{e^\tau} d\sigma_g \right) d\tau, \end{aligned} \quad (5.3)$$

where in the third equality we used identity (3.6) and the definition of  $\Phi_p$  given by formula (3.9). By the Fundamental Theorem of Calculus, we have that if the function

$$\tau \longmapsto \frac{\Phi_p(\tau)}{e^\tau} + (p-1) \int_{\{\varphi = \tau\}} \frac{|\nabla \varphi|_g^{p-1} H_g}{e^\tau} d\sigma_g$$

is continuous, then  $\Phi_p$  is differentiable. Since we have already discussed in Subsection 5.1 the continuity of  $s \mapsto \Phi_p(s)$ , we only need to discuss the continuity of the assignment

$$\tau \longmapsto \int_{\{\varphi = \tau\}} \frac{|\nabla \varphi|_g^{p-1} H_g}{e^\tau} d\sigma_g = \int_{\{\varphi > \tau\}} \frac{|\nabla \varphi|_g^{p-3} \left( |\nabla^2 \varphi|_g^2 + (p-3) |\nabla |\nabla \varphi|_g|^2_g \right)}{e^\varphi} d\mu_g. \quad (5.4)$$

We note that the above equality follows from the integral identity (4.8) in Proposition 4.2, which is in force under the hypotheses of Theorem 3.2-(ii). In analogy with (5.2), it is natural to set

$$\bar{\mu}_g^{(p)}(E) := \int_E \frac{|\nabla \varphi|_g^{p-3} \left( |\nabla^2 \varphi|_g^2 + (p-3) |\nabla |\nabla \varphi|_g|^2_g \right)}{e^\varphi} d\mu_g,$$

for every  $\mu_g$ -measurable set  $E \subseteq \{\varphi > \varepsilon\}$ . From identity (4.8) we have that the measure  $\bar{\mu}_g^{(p)}$  is finite for every  $p \geq 2 - 1/(n-1)$ . Hence, using [4, Proposition 2.6] and the same reasoning as in Subsection 5.1, we deduce that the assignment (5.4) is continuous. In turn, we obtain the differentiability of  $\Phi_p$ . Finally, using (4.8) and (5.3), a direct computation shows that

$$\begin{aligned} \Phi'_p(s) &= -(p-1) \int_{\{\varphi = s\}} |\nabla \varphi|_g^{p-1} H_g d\sigma_g \\ &= -(p-1) e^s \int_{\{\varphi > s\}} \frac{|\nabla \varphi|_g^{p-3} \left( |\nabla^2 \varphi|_g^2 + (p-3) |\nabla |\nabla \varphi|_g|^2_g \right)}{e^\varphi} d\mu_g. \end{aligned} \quad (5.5)$$

The monotonicity and the rigidity statements in Theorem 3.2-(ii) are now consequences of Proposition 4.2.



## APPENDIX: STRUCTURAL PROPERTIES OF THE CRITICAL SET

In this appendix we state and prove some properties of the set  $\text{Crit}(\varphi)$  given by the critical points of  $\varphi$ , where  $\varphi$  is a solution to problem (3.3). These properties were mentioned and used in the proof of the integral identities in Section 4. We also give some estimates for the quantity  $\langle \nabla |\nabla \varphi|_g^{p-1} |n_g \rangle_g$  at the boundary of a suitable small neighbourhood of  $\text{Crit}(\varphi)$ , where  $n_g$  is the normal direction. We restrict the analysis to the nontrivial case where  $2 - 1/(n-1) \leq p < 2$  (cfr. the proof of Proposition 4.2). More in general, the results gathered in this section hold for  $1 < p < 2$ . However, we recall that the lower bound  $p \geq 2 - 1/(n-1)$  is needed in Proposition 4.2 to ensure the nonnegativity of the right-hand side in (4.8). It is also worth noticing that, even though the results of this section are applied to solutions of problem (3.3), they hold for arbitrary  $g$ -harmonic functions with compact level sets.

We start by recalling that the set of the critical values of a real analytic function is discrete, as proven in [25]. Hence, if  $\varphi$  is a real analytic function defined in  $(M, g)$  with compact level sets, then

$$\text{Crit}(\varphi) \cap \{\varphi = s\} = \text{Crit}(\varphi) \cap B_\delta(\{\varphi = s\}), \quad (\text{A-1})$$

for every  $\delta > 0$  sufficiently small. Here and throughout this section we use the notation

$$B_\delta(E) := \{x \in M : \text{dist}_g(x, E) < \delta\}, \quad (\text{A-2})$$

for every subset  $E$  of  $M$ . The following two theorems give a quite detailed information about the behaviour of a real analytic function around its critical set and about the structure of the critical set itself. For the proof of both theorems, we refer the reader to [21, Section 6].

**Theorem A.1** (Weierstrass Preparation Theorem). *Let  $\psi = \psi(x^1, \dots, x^n)$  be a nontrivial (i.e., non identically zero) real analytic function defined in a neighbourhood of the origin of the flat  $n$ -dimensional Euclidean space which is vanishing at the origin. By analyticity, we can suppose, after a normalization, that the function  $x^n \mapsto \psi(0, \dots, 0, x^n)$  is not identically zero. Then, in a possibly smaller neighbourhood of the origin,  $\psi$  can be written as  $\psi = HU$ , where  $U$  is a function which never vanishes and  $H$  is a Weierstrass polynomial, namely*

$$H(x^1, \dots, x^n) = (x^n)^m + A_1(x^1, \dots, x^{n-1})(x^n)^{m-1} + \dots + A_{m-1}(x^1, \dots, x^{n-1})x^n + A_m(x^1, \dots, x^{n-1}),$$

for some integer  $m \geq 1$ . Here, for every  $i = 1, \dots, m$ , the coefficient  $A_i$  is an analytic function vanishing at  $(x^1, \dots, x^{n-1}) = (0, \dots, 0)$ .

**Theorem A.2** (Lojasiewicz's Structure Theorem). *Let  $\varphi$  be a real analytic function in  $M$ . For every  $x_0 \in \{\nabla \varphi = 0\}$ , there exists  $\delta > 0$  such that*

$$\{x \in B_\delta(x_0) : \nabla \varphi(x) = 0\} = V^{n-1} \cup \dots \cup V^1 \cup V^0,$$

where for every  $k \in \{1, \dots, n-1\}$ , either  $V^k = \emptyset$  or there exists a finite number  $n_k$  of  $k$ -dimensional disjoint analytic submanifolds  $M_1^k, \dots, M_{n_k}^k$  such that

$$V^k = \bigsqcup_{j=1}^{n_k} M_j^k.$$

Moreover, we have that either  $V^0 = \emptyset$  or  $V^0 = \{x_0\}$ . Furthermore, for every  $k \in \{1, \dots, n-1\}$ , if  $V^k \neq \emptyset$ , then the following stratification property holds

$$\overline{V^k} \supseteq (V^{k-1} \cup \dots \cup V^0). \quad (\text{A-3})$$

In the following we apply these classical theorems to  $g$ -harmonic functions, for which the Hausdorff dimension of the critical set is known to be bounded above by  $n-2$ . Hence, from now on the index  $k$  of the above theorem will range between 0 and  $n-2$ . Building on this fact, we construct a suitable small neighbourhood  $U_\varepsilon$  of  $\text{Crit}(\varphi)$ , called *stratified tubular neighbourhood*. Since our ultimate goal is to ensure that the flux of  $\nabla |\nabla \varphi|_g^{p-1}$  through the boundary of  $U_\varepsilon$  is infinitesimal with respect to  $\varepsilon$  (see the second estimate in (A-9)), our construction needs to be accurate enough to provide both an estimate of the area of  $\partial U_\varepsilon$  and an acceptable upper bound for the quantity  $\langle \nabla |\nabla \varphi|_g^{p-1} |n_g \rangle_g$  on  $\partial U_\varepsilon$ .

**Proposition A.3.** *Given a  $g$ -harmonic function  $\varphi$  in  $M$  with compact level sets, consider the set*

$$\mathcal{C}_s := \{\varphi = s\} \cap \text{Crit}(\varphi), \quad (\text{A-4})$$

for some  $s \in \varphi(M)$ . For every  $\varepsilon > 0$  sufficiently small and for some integer  $k \leq n-2$ , there exists a family  $\{\varepsilon_i\}_{i=0}^k$  of small parameters fulfilling

$$0 < \varepsilon_k < \dots < \varepsilon_1 < \varepsilon_0 = \varepsilon, \quad (\text{A-5})$$

other two families of positive constants  $\{c_i\}_{i=0}^k$  and  $\{d_i\}_{i=0}^k$ , and a stratified tubular neighbourhood  $U_\varepsilon$  of  $\mathcal{C}_s$  satisfying the following properties:

(i)  $\partial U_\varepsilon = \bigsqcup_{i=0}^k S_{\varepsilon_i}^i \sqcup N$ , where

$$\mathcal{H}^{n-1}(N) = 0 \quad \text{and} \quad \mathcal{H}^{n-1}(S_{\varepsilon_i}^i) \leq d_i \varepsilon_i \quad \text{for } i = 0, \dots, k; \quad (\text{A-6})$$

(ii) if  $\xi \in S_{\varepsilon_i}^i$ , for some  $i = 0, \dots, k$ , then

$$|\langle \nabla |\nabla \varphi|_g^{p-1} | n_g \rangle_g|(\xi) \leq c_i \varepsilon_i^{\frac{p-1}{2}-1}, \quad (\text{A-7})$$

for every  $1 < p < 2$ . Here,  $n_g$  is the (outer) unit normal of  $U_\varepsilon$ , which is everywhere defined on  $S_{\varepsilon_i}^i$ , for every  $i = 0, \dots, k$ , and in turn  $\mathcal{H}^{n-1}$ -a.e. defined on  $\partial U_\varepsilon$ .

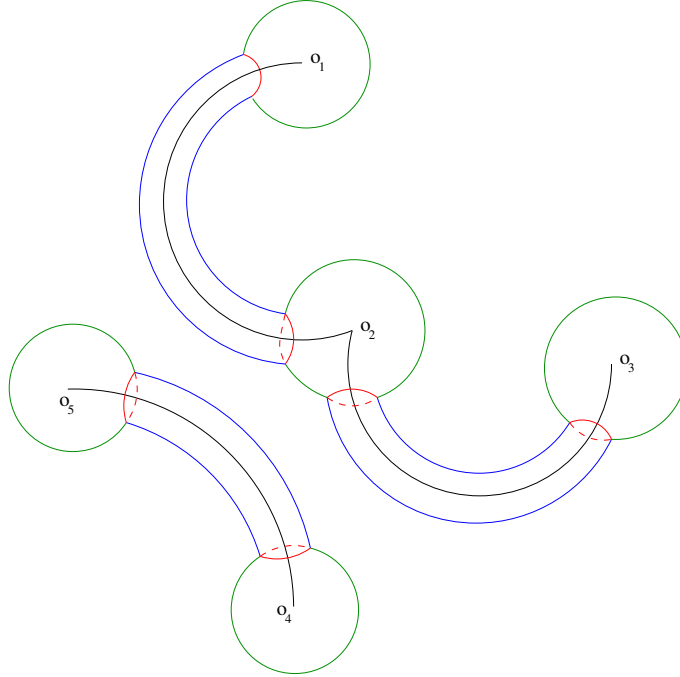


FIGURE 2. Sketch of the set  $\mathcal{C}_s$  (black lines) and of the stratified tubular neighbourhood  $U_\varepsilon$ . Note that in this picture  $\mathcal{C}_s = V^0 \sqcup V^1$ , where  $V^0 = \{O_1, \dots, O_5\}$  and  $V^1$  is the disjoint union of the arcs connecting the points  $O_j$ 's. Using the notation of Proposition A.3, we have that  $\partial U_\varepsilon$  is given by the disjoint union of  $S_\varepsilon^0$  (green spheres),  $S_{\varepsilon_1}^1$  (blue tubes), and  $N$  (red circles).

The reason of the name given to the neighbourhood  $U_\varepsilon$  can be partially suggested by the decomposition of  $\partial U_\varepsilon$  in point (i) and will be further clarified in the proof. Note that, by the absolute continuity with respect of  $\mathcal{H}^{n-1}$  of the measure induced by  $g$  on  $S_{\varepsilon_i}^i$ , the estimates in (A-6) imply

$$\int_{S_{\varepsilon_i}^i} d\sigma_g \leq e_i \varepsilon_i, \quad \text{for } i = 0, \dots, k, \quad (\text{A-8})$$

for some constants  $e_i > 0$ . In particular, the previous proposition implies that

$$\int_{\partial U_\varepsilon} |\langle \nabla |\nabla \varphi|_g^{p-1} | n_g \rangle_g| d\sigma_g = \sum_{i=0}^k \int_{S_{\varepsilon_i}^i} |\langle \nabla |\nabla \varphi|_g^{p-1} | n_g \rangle_g| d\sigma_g \leq \sum_{i=0}^k c_i e_i \varepsilon_i^{\frac{p-1}{2}} \leq \varepsilon^{\frac{p-1}{2}} \sum_{i=0}^k c_i e_i,$$

where in the equality we have used the decomposition of  $\partial U_\varepsilon$  into  $\partial U_\varepsilon = \bigsqcup_{i=0}^k S_\varepsilon^i \sqcup N$  and the first of (A-6), and in the inequalities we have used (A-5), (A-7), and (A-8). The derived estimate is the content of the following corollary.

**Corollary A.4.** *Given a  $g$ -harmonic function  $\varphi$  in  $M$  with compact level sets, let  $\mathcal{C}_s$  be the set defined in (A-4), for some  $s \in \varphi(M)$ . For every  $\varepsilon > 0$  sufficiently small, there exists a stratified tubular neighbourhood  $U_\varepsilon$  of  $\mathcal{C}_s$  such that the (outer) unit normal  $\mathbf{n}_g$  is defined  $\mathcal{H}^{n-1}$ -a.e. on  $\partial U_\varepsilon$  and such that*

$$\mathcal{H}^{n-1}(\partial U_\varepsilon) \leq d\varepsilon, \quad \int_{\partial U_\varepsilon} |\langle \nabla |\nabla \varphi|_g^{p-1} | \mathbf{n}_g \rangle_g| d\sigma_g \leq c\varepsilon^{\frac{p-1}{2}}, \quad (\text{A-9})$$

for every  $1 < p < 2$ , for some constants  $d > 0$  and  $c > 0$  independent of  $\varepsilon$ .

*Proof of Proposition A.3.* Note first that, by compactness, the set  $\mathcal{C}_s$  is made of a finite number of connected components. To simplify the notation, let us make throughout the proof the non restrictive assumption that  $\mathcal{C}_s$  is connected. Using the properness of  $\varphi$  and the preliminary observation (A-1), combined with Theorem A.2, one can decompose the set  $\mathcal{C}_s$  as

$$\mathcal{C}_s = V^k \cup \dots \cup V^0, \quad k := \max \left\{ i \in \{0, \dots, n-2\} : V^i \neq \emptyset \right\},$$

where for every  $i \in \{1, \dots, k\}$ , either  $V^i = \emptyset$  or there exists a finite family of  $i$ -dimensional disjoint submanifolds  $M_1^i, \dots, M_{n_i}^i$  such that

$$V^i = \bigsqcup_{l=1}^{n_i} M_l^i. \quad (\text{A-10})$$

Also, we have that either  $V^0 = \emptyset$  or  $V^0$  is the collection of  $n_0$  distinct points of  $M$ . We recall that the bound  $k \leq (n-2)$  is due to the fact that the Hausdorff dimension of  $\mathcal{C}_s$  is bounded by  $(n-2)$ , which comes from the  $g$ -harmonicity of  $\varphi$ . Notice that, by the stratification property (A-3), we have that

$$\bigsqcup_{l=1}^{n_i} \partial M_l^i = \partial V^i = V^0 \sqcup \dots \sqcup V^{i-1}, \quad \text{for every } i = 1, \dots, k. \quad (\text{A-11})$$

where the first identity is a trivial consequence of (A-10). We are now in the position to construct the neighbourhood  $U_\varepsilon$  of  $\mathcal{C}_s$ . Let us suppose that  $V^0 \neq \emptyset$ , otherwise we can start the construction from the stratum  $V^h$ , where

$$h := \min \left\{ i \in \{0, \dots, k\} : V^i \neq \emptyset \right\}.$$

Let  $\varepsilon = \varepsilon_0 > 0$  be such that  $\overline{B_\varepsilon}(\eta_1) \cap \overline{B_\varepsilon}(\eta_2) = \emptyset$ , for every  $\eta_1, \eta_2 \in V^0$  with  $\eta_1 \neq \eta_2$ . We remark that in the case where  $h > 0$  the key point for starting the construction is that  $V^h$  is made of a (finite) number of connected components  $M_l^h$  having positive distance from each other. Consider now  $V^1$  and observe from (A-11) that for every  $l, m \in \{1, \dots, n_1\}$  with  $l \neq m$  we have

$$(\overline{M_l^1} \setminus B_\varepsilon(V^0)) \cap (\overline{M_m^1} \setminus B_\varepsilon(V^0)) = \emptyset$$

and in turn there exists  $0 < \varepsilon_1 < \varepsilon$  such that

$$(\overline{B_{\varepsilon_1}}(M_l^1) \setminus B_\varepsilon(V^0)) \cap (\overline{B_{\varepsilon_1}}(M_m^1) \setminus B_\varepsilon(V^0)) = \emptyset.$$

In particular, for every  $\xi \in \partial(B_\varepsilon(V^0) \cup B_{\varepsilon_1}(V^1)) \setminus (\partial B_\varepsilon(V^0) \cap \partial B_{\varepsilon_1}(V^1))$ , there exists a unique  $l \in \{1, \dots, n_1\}$  and a unique  $\zeta \in M_l^1$  such that  $\text{dist}_g(\xi, \mathcal{C}_s) = \text{dist}_g(\xi, \zeta)$ . As a consequence, one has that on the set

$$\partial(B_\varepsilon(V^0) \cup B_{\varepsilon_1}(V^1)) \setminus (\partial B_\varepsilon(V^0) \cap \partial B_{\varepsilon_1}(V^1)),$$

which is  $\mathcal{H}^{n-1}$ -a.a. of the boundary of  $B_\varepsilon(V^0) \cup B_{\varepsilon_1}(V^1)$ , the normal to  $B_\varepsilon(V^0) \cup B_{\varepsilon_1}(V^1)$  is everywhere defined. We refer the reader to Figure 5.2 for a sketch of these sets. Setting

$$S_\varepsilon^0 := \partial B_\varepsilon(V^0) \setminus \overline{B_{\varepsilon_1}}(V^1), \quad S_{\varepsilon_1}^1 := \partial B_{\varepsilon_1}(V^1) \setminus \overline{B_\varepsilon}(V^0), \quad N := \partial B_\varepsilon(V^0) \cap \partial B_{\varepsilon_1}(V^1),$$

we have that point (i) is fulfilled in the case where  $k = 1$ . In particular, since  $S_\varepsilon^0$  is an open subset of a disjoint union of spheres of radius  $\varepsilon$  and  $S_{\varepsilon_1}^1$  is an open subset of a disjoint union of pieces of boundaries of  $\varepsilon_1$ -tubular neighbourhood of 1-dimensional manifolds, we obtain the estimates

$$\mathcal{H}^{n-1}(S_\varepsilon^0) \leq d_0 \varepsilon^{n-1}, \quad \mathcal{H}^{n-1}(S_{\varepsilon_1}^1) \leq d_1 \varepsilon_1^{n-2}.$$

When  $k > 1$ , it is enough repeating a similar construction for the sets  $V^2, \dots, V^k$ , leading to the estimate

$$\mathcal{H}^{n-1}(S_{\varepsilon_i}^i) \leq d_i \varepsilon_i^{n-i-1}, \quad \text{for } i = 0, \dots, k.$$

Using the fact that  $k \leq (n-2)$ , we thus get (A-6). This construction yields the definition of the *stratified tubular neighbourhood*

$$U_\varepsilon := \bigcup_{i=0}^k B_{\varepsilon_i}(V^i), \quad \text{with } \varepsilon_{i+1} < \varepsilon_i \text{ for every } i = 0, \dots, k-1,$$

which fulfills point (i).

To prove point (ii), observe first that from the above construction it follows that for every  $\xi \in \partial U_\varepsilon \setminus N$  the normal to  $U_\varepsilon$  at  $\xi$  is well defined. Also, if  $\xi \in S_{\varepsilon_i}^i$ , there exist a unique  $l \in \{1, \dots, n_i\}$  and a unique  $\zeta \in M_l^i$  such that

$$\text{dist}_g(\xi, C_s) = \text{dist}_g(\xi, \zeta). \quad (\text{A-12})$$

To check that the upper bound (A-7) holds, note that estimating  $|\langle \nabla |\nabla \varphi|_g^{p-1} | n_g \rangle_g|(\xi)$ , when  $p$  is varying between 1 and 2, is equivalent to estimating

$$\frac{|\langle \nabla |\nabla \varphi|_g^2 | n_g \rangle_g|}{(|\nabla \varphi|_g^2)^{1-\alpha}}(\xi), \quad \text{with } 0 < \alpha < \frac{1}{2}.$$

Now, for a given point  $\xi \in S_{\varepsilon_i}^i$  we let  $\zeta \in V^i$  be the point realising (A-12), with  $i \in \{1, \dots, k\}$  and  $k \leq n-2$ . Choosing  $\varepsilon$  small enough, we have that there exist an open neighbourhood  $\Omega_{\varepsilon_i}$  of both  $\zeta$  and  $\xi$ , a real number  $\delta_i = \delta_i(\varepsilon_i) > 0$  and an analytic diffeomorphism  $f : \Omega_{\varepsilon_i} \subset M \rightarrow B_{\delta_i}^i(0) \times B_{2\varepsilon_i}^{n-i}(0) \subset \mathbb{R}^n$ , with  $B_{\delta_i}^i(0) \subset \mathbb{R}^i$  and  $B_{2\varepsilon_i}^{n-i}(0) \subset \mathbb{R}^{n-i}$ , such that  $f(\zeta) = (0, \dots, 0)$ ,  $f(\xi) = (0, \dots, 0, \varepsilon_i)$  and

$$f(V^i \cap \Omega_{\varepsilon_i}) = B_{\delta_i}^i(0) \times \{0\} \subset \mathbb{R}^i \times \mathbb{R}^{n-i}.$$

In any case, we have that the function

$$\psi := |\nabla \varphi|_g^2 \circ f^{-1} : B_{\delta_i}^i(0) \times B_{2\varepsilon_i}^{n-i}(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

is a real analytic function such that  $\psi(0, \dots, 0, x^n)$  is not identically zero. Note that, due to compactness, there exists a constant  $C > 0$  independent of  $\xi \in S_{\varepsilon_i}^i$ , of  $i \in \{0, \dots, k\}$ , and of  $\varepsilon_i$ , such that

$$\frac{|\langle \nabla |\nabla \varphi|_g^2 | n_g \rangle_g|}{(|\nabla \varphi|_g^2)^{1-\alpha}}(\xi) \leq C \frac{\left| \frac{\partial \psi}{\partial x^n} \right|}{\psi^{1-\alpha}}(0, \dots, 0, \varepsilon_i). \quad (\text{A-13})$$

Since  $\psi(0, \dots, 0, x^n)$  is not identically zero in  $B_{\delta_i}^i(0) \times B_{2\varepsilon_i}^{n-i}(0) \subset \mathbb{R}^n$ , from Theorem A.1 we have that  $\psi = HU$ , where  $U(x) \neq 0$  for every  $x \in B_{\delta_i}^i(0) \times B_{2\varepsilon_i}^{n-i}(0)$  and  $H$  is Weierstrass polynomial in  $x^n$ . In particular, we have that

$$\frac{\partial \psi}{\partial x^n}(0, \dots, 0, \varepsilon_i) = \varepsilon_i^{m-1} U(0, \dots, 0, \varepsilon_i) + \varepsilon_i^m \frac{\partial U}{\partial x^n}(0, \dots, 0, \varepsilon_i),$$

for some integer  $m \geq 0$ , and in turn that

$$\begin{aligned} \frac{\left| \frac{\partial \psi}{\partial x^n} \right|}{\psi^{1-\alpha}}(0, \dots, 0, \varepsilon_i) &= \frac{\left| \varepsilon_i^{m-1} U(0, \dots, 0, \varepsilon_i) + \varepsilon_i^m \frac{\partial U}{\partial x^n}(0, \dots, 0, \varepsilon_i) \right|}{\left| \varepsilon_i^m U(0, \dots, 0, \varepsilon_i) \right|^{1-\alpha}} \\ &= \varepsilon_i^{m\alpha-1} \frac{\left| U(0, \dots, 0, \varepsilon_i) + \varepsilon_i \frac{\partial U}{\partial x^n}(0, \dots, 0, \varepsilon_i) \right|}{\left| U(0, \dots, 0, \varepsilon_i) \right|^{1-\alpha}} \leq C \varepsilon_i^{\frac{p-1}{2}-1}, \end{aligned}$$

where in the last passage we have bounded  $\varepsilon_i^{m\alpha}$  by  $\varepsilon_i^\alpha$  and replaced  $\alpha$  by  $(p-1)/2$ . Also, note that in the last inequality the constant is uniform in  $\xi \in S_{\varepsilon_i}^i$ , in  $i \in \{0, \dots, k\}$ , and in  $\varepsilon_i > 0$ , because of compactness and because in the terms  $U(0, \dots, 0, \varepsilon_i)$  and  $\partial U / \partial x^n(0, \dots, 0, \varepsilon_i)$  nothing but derivatives of  $\psi$  appear, up to a certain finite order. The last inequality, coupled with (A-13), gives estimate (A-7).

□

**Acknowledgements.** *V. A. has been partially funded by the European Research Council / ERC Advanced Grant n° 340685. L. M. has been partially supported by the Italian project FIRB 2012 “Geometria Differenziale e Teoria Geometrica delle Funzioni”. The authors are members of Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), which is part of the Istituto Nazionale di Alta Matematica (INdAM), and partially funded by the GNAMPA project “Principi di fattorizzazione, formule di monotonia e disuguaglianze geometriche”.*

## REFERENCES

- [1] V. Agostiniani and L. Mazziere. On the geometry of the level sets of bounded static potentials. arXiv:1504.04563.
- [2] V. Agostiniani and L. Mazziere. Riemannian aspects of potential theory. *J. Math. Pures Appl.*, 104(3):561 – 586, 2015.
- [3] A. D. Alexandrov. A characteristic property of spheres. *Annali di Matematica Pura ed Applicata*, 58(1):303–315, 1962.
- [4] L. Ambrosio, G. Da Prato, and A. Mennucci. *Introduction to measure theory and integration*. Lecture notes 10. Edizioni della Normale, 2011.
- [5] C. Bianchini and G. Ciraolo. *A Note on an Overdetermined Problem for the Capacitary Potential*, pages 41–48. Springer International Publishing, Cham, 2016.
- [6] V. Bour and G. Carron. Optimal integral pinching results. arXiv:1203.0384.
- [7] S. Brendle, P.-K. Hung, and M.-T. Wang. A Minkowski inequality for hypersurfaces in the Anti-de Sitter-Schwarzschild manifold. *Communications on Pure and Applied Mathematics*, 69(1):124–144, 2016.
- [8] J. Cheeger, A. Naber, and D. Valtorta. Critical sets of elliptic equations. arXiv:1207.4236v3.
- [9] B.-Y. Chen. On a theorem of Fenchel-Borsuk-Willmore-Chern-Lashof. *Mathematische Annalen*, 194(1):19–26, 1971.
- [10] B.-Y. Chen. On the total curvature of immersed manifolds, I: An inequality of Fenchel-Borsuk-Willmore. *American Journal of Mathematics*, 93(1):148–162, 1971.
- [11] T. H. Colding. New monotonicity formulas for Ricci curvature and applications. I. *Acta Mathematica*, 209(2):229–263, 2012.
- [12] T. H. Colding and W. P. Minicozzi. Monotonicity and its analytic and geometric implications. *Proceedings of the National Academy of Sciences*, 110(48):19233–19236, 2013.
- [13] T. H. Colding and W. P. Minicozzi. Ricci curvature and monotonicity for harmonic functions. *Calculus of Variations and Partial Differential Equations*, 49(3):1045–1059, 2014.
- [14] G. Crasta, I. Fragalà, and F. Gazzola. On a long-standing conjecture by Pólya-Szegő and related topics. *Z. Angew. Math. Phys.*, 56(5):763–782, 2005.
- [15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
- [16] A. Farina, L. Mari, and E. Valdinoci. Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds. *Comm. Partial Differential Equations*, 38(10):1818–1862, 2013.
- [17] R. Hardt, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili. Critical sets of solutions to elliptic equations. *J. Differential Geom.*, 51(2):359–373, 1999.
- [18] R. Hardt and L. Simon. Nodal sets for solutions of elliptic equations. *J. Differential Geom.*, 30(2):505–522, 1989.
- [19] G. Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.
- [20] O. D. Kellogg. *Foundations of potential theory*. Springer, 1967.
- [21] S. G. Krantz and H. R. Parks. *A primer of real analytic functions*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.
- [22] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159.
- [23] W. Reichel. Radial symmetry for elliptic boundary-value problems on exterior domains. *Arch. Rational Mech. Anal.*, 137(4):381–394, 1997.
- [24] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [25] J. Souček and V. Souček. Morse-Sard theorem for real-analytic functions. *Comment. Math. Univ. Carolin.*, 13(1):45–51, 1972.
- [26] T. J. Willmore. Mean curvature of immersed surfaces. *An. Ști. Univ. “Al. I. Cuza” Iași Sect. I a Mat. (N.S.)*, 14:99–103, 1968.
- [27] J. Xiao.  $p$ -capacity vs surface-area. arXiv:1506.03827.

V. AGOSTINIANI, SISSA, VIA BONOMEA 265, 34136 TRIESTE, ITALY  
*E-mail address:* `vagostin@sisssa.it`

L. MAZZIERI, UNIVERSITÀ DEGLI STUDI DI TRENTO, VIA SOMMARIVE 14, 38123 POVO (TN), ITALY  
*E-mail address:* `lorenzo.mazziere@unitn.it`